Likelihood ratio test.

Neyman Pearson lemma based on the magnitude of the ratio of two probability distribution functions. It encompasses the test between simple hypothesis against simple alternative hypothesis.

Likelihood ratio testing procedure is a general method of test construction. By this process one can test any hypothesis (simple or composite alternative)

Test Construction

Suppose that the null hypothesis specifies that θ lies in a particular set of possible values, say Θ_0, i.e., $H_0: \theta \in \Theta_0$; the alternative hypothesis specifies that Θ lies in another set of possible values Θ_A, which does not overlap Θ_0, i.e: $H_1: \theta \in \Theta_A$ such that $\Theta = \Theta_0 \cup \Theta_A$. Either or both of the hypotheses H_0 and H_1 can be compositional.

Let $L(\Theta_0)$ be the supremum (maximum) of the likelihood function for all $\theta \in \Theta_0$

$$\cancel{L(\theta} \qquad L(\hat{\Theta}_0) = \max_{\theta \in \Theta_0} L(\theta).$$

$L(\hat{\Theta}_0)$ represents the best explanation for the observed data for $\theta \in \Theta_0$.
Similarly, $L(\hat{\Theta}) = \max_{\theta \in \Theta} L(\theta)$ represents the best explanation for the observed data for all $\theta \in \Theta$.
According to the principle of maximum likelihood, the likelihood equation for estimating any parameter θ_i is given by

$$\frac{\partial L}{\partial \hat{\theta}_i} = 0 \quad (i = 1, 2, \cdots, k)$$

If $L(\hat{H}_0) = L(\hat{\Theta})$, then a best explanation for the observed data can be found inside Θ_0 and we should not reject the null hypothesis $H_0: \theta \in \Theta_0$.
However, if $L(\hat{\Theta}_0) < L(\hat{\Theta})$, then the best explanation for the observed data could be found inside Θ_A and we reject Ho in favour of H_1.

Then the criterion for the likelihood ratio test is defined as the quotient of these two maxima and is given by

$$\lambda = \lambda(x_1, x_2, \cdots, x_n)$$
$$= \frac{L(\hat{\Theta}_0)}{L(\hat{\Theta})}$$
$$= \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)}$$

The quantity λ is a function of the sample observations only and does not involve parameters. Thus λ being a function of the random variables, is also a random variable. Obviously $\lambda > 0$. Further,

$$\Theta_0 \subset \Theta \Rightarrow L(\Theta_0) \leq L(\Theta) \Rightarrow \lambda \leq 1$$

$\therefore$ Hence $\boxed{0 \leq \lambda \leq 1}$.

The critical region for testing H_0 (against H_1) is an interval; $0 < \lambda \leq \lambda_0$ where λ_0 is some number (<1) determined by the distribution of λ where

$$P(\lambda < \lambda_0 | H_0) = \alpha$$

Look at the rejection criterion: it is left tail rejection. Why?

✓ However, if we are dealing with large samples, an asymptotic distribution of λ can be deduced under H_0.

$$-2 \log_e \lambda \sim \chi^2_r$$ [r degrees of freedom = difference in the number of parameters between the two models]

✓ For a simple H_0 vs. simple H_1 test, LR test leads to the same test as given by Neyman-Pearson lemma.

Test of the mean of a normal population

$X_1, X_2, \ldots, X_n$ be a random sample of size n from Normal (μ, σ^2) where μ unknown and σ^2 known.

Consider $H_0: \mu = \mu_0$ (simple null). $\boxed{\text{Here } \theta = \mu.}$

<u>Case I</u> $H_1: \mu \neq \mu_0$.

Here $\Theta = \{\mu \mid -\infty < \mu < \infty\}$; $\Theta_0 = \{\mu \mid \mu = \mu_0\}$

Now, $L(\mu, \sigma^2) = \frac{1}{(\sigma\sqrt{2\pi})^n} \exp[-\Sigma(x_i - \mu)^2/2\sigma^2]$

$L(\mu_0, \sigma^2) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\Sigma(x_i-\mu_0)^2/2\sigma^2}$

But under Θ $L(\mu, \sigma^2)$ will be maximum when we choose $\hat{\mu} = \bar{x}$ (sample mean).

$$\sup_{\theta \in \Theta} L(\hat{\mu}, \sigma^2) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\Sigma(x_i - \bar{x})^2/2\sigma^2}$$

Thus

$$\lambda = \frac{\sup_{\theta \in \Theta_0} L(\hat{\Theta}_0)}{\sup_{\theta \in \Theta} L(\hat{\Theta})} = e^{-\frac{1}{2}\Sigma(x_i-\mu_0)^2/2\sigma^2 + \frac{1}{2\sigma^2}\Sigma(x_i-\bar{x})^2}$$

$$= \exp^{-\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2}.$$

Now reject H_0, if $\lambda < \lambda_0$

$$\equiv -\frac{n}{2\sigma^2}(\bar{x}-\mu_0)^2 < \lambda_1 \text{ (say)}$$

$$\equiv \left| \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} \right| > \lambda_2$$

Critical region

$$W = \left\{ \underline{x} : \left| \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} \right| > \lambda_2 \right\}$$

where λ_2 is such that

$$P_{H_0}(W) = \alpha$$

$$\Rightarrow P_{H_0}\left[\left| \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} \right| > \lambda_2 \right] = \alpha.$$

Now $\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} \sim N(0,1)$ under H_0

So $\lambda_2 = \tau_{\alpha/2}$.

Thus,

$$W = \left\{ \underline{x} \,\middle|\, \left| \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} \right| > \tau_{\alpha/2} \right\}$$

<u>case II</u> $H_1: \mu > \mu_0$.

Here $\Theta = \{ \mu \mid \mu_0 \le \mu < \infty \}$

Maximum likelihood estimate for θ for $\theta \in \Theta$ is

$$\hat{\theta} = \begin{cases} \bar{x} & \text{if } \bar{x} \ge \mu_0 \\ \mu_0 & \text{if } \bar{x} < \mu_0. \end{cases}$$

$$\therefore \sup_{\theta \in \Theta} L(\theta / x_1, x_2, \ldots, x_n) = \begin{cases} \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum(x_i-\bar{x})^2} & \text{if } \bar{x} \ge \mu_0 \\ \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu_0)^2} & \text{if } \bar{x} < \mu_0 \end{cases}$$

$$\therefore \lambda = \begin{cases} e^{-n/2\sigma^2(\bar{x}-\mu_0)^2} & \text{if } \bar{x} \ge \mu_0. \\ 1 & \text{if } \bar{x} < \mu_0. \end{cases}$$

So the rejection region $\lambda < \lambda_0$ is equivalent to the condition.

$$\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} > \lambda_1$$

$$\equiv W_1 = \left\{ \underline{x} \,\middle|\, \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} > \lambda_1 \right\}$$

where λ_1 is such that

$$P_{\theta_0}\left\{ \frac{\sqrt{n}(\bar{X}-\mu_0)}{\sigma} > \lambda_1 \right\} = \alpha.$$

So, $\lambda_1 = \tau_\alpha$

So, $W_1 = \left\{ \underline{x} \,\middle|\, \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} > \tau_\alpha \right\}$.

<u>case 3</u> $H_1: \mu < \mu_0$

$$\Theta = \{ \mu \mid -\infty < \mu \le \mu_0 \}$$

$$\hat{\theta} = \begin{cases} \bar{x} & \text{if } \bar{x} \le \mu_0 \\ \mu_0 & \text{if } \bar{x} > \mu_0. \end{cases}$$

Proceeding as Case 2, the critical region.

$$W_2 = \left\{ \underline{x} \,\middle|\, \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} < -\tau_\alpha \right\}$$